Homework 2: The Intertemporal Capital Asset Pricing Model

Instructions

Answer each part carefully. Show all intermediate steps. Unless otherwise stated, all processes are adapted and satisfy the usual regularity conditions. Asset excess returns are measured relative to a constant risk-free rate r.

Setup (Given)

There are n risky assets with prices $\{P_{it}\}_{i=1}^{n}$. A (possibly scalar) state variable z_t may affect drifts and diffusions. Throughout, Brownian shocks may be multi-dimensional and correlated.

$$\frac{\mathrm{d}P_{it}}{P_{it}} = \mu_i(z_t)\,\mathrm{d}t + \sigma_i(z_t)^\top \mathrm{d}B_t, \qquad i = 1, \ldots, n, \tag{1}$$

$$\mathrm{d}z_t = a(z_t)\,\mathrm{d}t + b(z_t)^\top \mathrm{d}B_t, \tag{2}$$

where $\mathrm{d}B_t$ is a k-dimensional Brownian motion driving *both* asset returns and z_t. Denote the $n \times n$ instantaneous covariance matrix by

$$\Sigma(z_t) \equiv \sigma(z_t)\sigma(z_t)^\top, \quad \text{with } \sigma(z_t) \equiv [\,\sigma_1(z_t) \ \cdots \ \sigma_n(z_t)\,]^\top.$$

Let $\omega_t \in \mathbb{R}^n$ be portfolio weights in risky assets, with the remainder in the risk-free asset, and C_t the consumption rate. Investor wealth W_t evolves under self-financing and consumption.

Exercises

1. Wealth dynamics (with and without z-dependence)

(a) Starting from (1), derive the SDE for wealth W_t when investing weights ω_t in risky assets and the remainder in the risk-free asset, while consuming at rate C_t. State your result in terms of $(\mu - r\mathbf{1})$, Σ, and W_t.

(b) Write the drift and diffusion components explicitly and show that

$$\mathrm{d}W_t = \Big(W_t\,\omega_t^\top(\mu - r\mathbf{1}) + rW_t - C_t\Big)\,\mathrm{d}t \ + \ W_t\,\omega_t^\top \sigma\,\mathrm{d}B_t.$$

(c) Specialize your expression to the *no-z* case in which μ_i and σ_i are constant.

Solution. (a)–(b) The portfolio return is $\omega_t^\top \frac{\mathrm{d}P_t}{P_t} + (1 - \mathbf{1}^\top \omega_t) r \, \mathrm{d}t$. With consumption C_t, self-financing implies

$$\mathrm{d}W_t = W_t \Big[\omega_t^\top (\mu - r\mathbf{1}) + r \Big] \mathrm{d}t - C_t \, \mathrm{d}t + W_t \, \omega_t^\top \sigma \, \mathrm{d}B_t,$$

which is the stated SDE. (c) With constant μ, σ, the same formula holds with time- and z-independent coefficients.

2. HJB without z: formulation and FOCs

Assume time-separable utility over consumption with discount rate $\rho > 0$, and value function $V(W)$ (stationary, no explicit t or z).

(a) Write the Hamilton–Jacobi–Bellman (HJB) equation for the investor who chooses (ω, C).

(b) Derive the first-order condition (FOC) for consumption and show that $U'(C) = V_W$.

(c) Derive the FOC for each portfolio weight ω_i and collect the resulting vector condition in matrix form.

Solution. (a) With no z and no t in V, the HJB is

$$\rho V(W) = \max_{\omega, C} \Big\{ U(C) + V_W \big(W \, \omega^\top (\mu - r\mathbf{1}) + rW - C \big) + \tfrac{1}{2} V_{WW} \, W^2 \, \omega^\top \Sigma \, \omega \Big\}.$$

(b) $U'(C) - V_W = 0 \Rightarrow U'(C) = V_W$. (c) The FOC in vector form is

$$V_W W (\mu - r\mathbf{1}) + V_{WW} W^2 \Sigma \, \omega = 0 \;\Rightarrow\; \omega = -\frac{V_W}{V_{WW} W} \, \Sigma^{-1} (\mu - r\mathbf{1}).$$

3. CRRA and the myopic (mean–variance) demand

Assume CRRA preferences and define relative risk aversion $\gamma \equiv -\dfrac{W V_{WW}}{V_W}$.

(a) Show that $-\dfrac{V_W}{V_{WW} W} = \dfrac{1}{\gamma}$.

(b) Conclude that the optimal risky-asset weights (no z) are

$$\omega^* \;=\; \frac{1}{\gamma} \, \Sigma^{-1} (\mu - r\mathbf{1}).$$

Solution. (a) From $\gamma = -\frac{W V_{WW}}{V_W}$ we obtain $-\frac{V_W}{V_{WW} W} = \frac{1}{\gamma}$. (b) Substitute into the FOC: $\omega^* = \frac{1}{\gamma} \Sigma^{-1} (\mu - r\mathbf{1})$.

4. From optimal weights to the CAPM

Let δ denote the *market* portfolio weights (aggregate of optimal policies, normalized so $\mathbf{1}^\top \delta = 1$). Assume $\delta \propto \Sigma^{-1}(\mu - r\mathbf{1})$.

(a) Show that there exists κ with $\mu - r\mathbf{1} = \kappa\, \Sigma\, \delta$.

(b) Let $r_m \equiv \delta^\top r$ denote the market return. Prove that

$$\kappa = \frac{\mu_m - r}{\operatorname{Var}(r_m)}.$$

(c) Deduce the classic CAPM relation

$$\mu_i - r \;=\; \beta_{im}\,(\mu_m - r), \qquad \beta_{im} \equiv \frac{\operatorname{Cov}(r_i, r_m)}{\operatorname{Var}(r_m)}.$$

Solution. (a) Since $\delta \propto \Sigma^{-1}(\mu - r\mathbf{1})$, there is κ with $\mu - r\mathbf{1} = \kappa\Sigma\delta$. (b) Multiply by $\delta^\top$: $\mu_m - r = \kappa\,\delta^\top\Sigma\delta = \kappa \operatorname{Var}(r_m)$, giving $\kappa = (\mu_m - r)/\operatorname{Var}(r_m)$. (c) Take the ith component: $\mu_i - r = \kappa e_i^\top \Sigma\delta = \frac{\mu_m - r}{\operatorname{Var}(r_m)} \operatorname{Cov}(r_i, r_m) = \beta_{im}(\mu_m - r)$.

5. Bringing back the state variable z: covariations

Define the z-covariation vector $\sigma_z \in \mathbb{R}^n$ by

$$\sigma_{iz} \;\equiv\; \frac{\operatorname{Cov}(\mathrm{d}r_i, \mathrm{d}z_t)}{\mathrm{d}t} = \sigma_i(z_t)^\top b(z_t), \qquad \sigma_z \equiv (\sigma_{1z}, \ldots, \sigma_{nz})^\top.$$

(a)) Verify that $\operatorname{Cov}(\mathrm{d}W_t, \mathrm{d}z_t)/\mathrm{d}t = W_t\,\omega_t^\top \sigma_z$.

(b)) State the economic meaning of σ_z: which shocks does it capture?

Solution. (a) $\mathrm{d}W_t$ has diffusion $W_t\,\omega_t^\top\sigma\,\mathrm{d}B_t$ and $\mathrm{d}z_t$ has diffusion $b^\top \mathrm{d}B_t$; hence $\operatorname{Cov}(\mathrm{d}W_t, \mathrm{d}z_t)/\mathrm{d}t = W_t\,\omega_t^\top \sigma b = W_t\,\omega_t^\top\sigma_z$. (b) σ_z measures each asset's instantaneous exposure to the shocks that drive z_t.

6. HJB with z: formulation and cross term

Let the value function now be $V(W, z)$ (stationary in calendar time).

(a) Write the HJB equation for this two-state problem, taking into account the drift and variance of W_t, the drift and variance of z_t, and the *cross* covariation between W_t and z_t.

(b) Identify the term that generates *hedging demand* (i.e., involves V_{Wz}).

Solution. (a) The HJB is

$$\rho V = \max_{\omega, C} \Big\{ U(C) + V_W\big(W\,\omega^\top(\mu - r\mathbf{1}) + rW - C\big) + \tfrac{1}{2} V_{WW} W^2\,\omega^\top\Sigma\omega + V_z\,a + \tfrac{1}{2} V_{zz}\,b^\top b + V_{Wz}\,W\,\omega^\top\sigma_z \Big\}.$$

(b) The hedging term is $V_{Wz}\,W\,\omega^\top\sigma_z$.

7. FOCs and optimal portfolio with hedging demand

(a) Derive the FOC for C and state the Euler condition.

(b) Derive the vector FOC for ω and solve for ω in closed form (matrix notation). *Hint:* collect the V_W, V_{WW}, and V_{Wz} terms.

(c) Using CRRA, show that the optimal weights can be written as the sum of a myopic and a hedging component:

$$\boxed{\omega^* = \underbrace{\frac{1}{\gamma}\,\Sigma^{-1}(\mu - r\mathbf{1})}_{\text{myopic}} \;+\; \underbrace{\frac{V_{Wz}}{\gamma V_W}\,\Sigma^{-1}\sigma_z}_{\text{hedging}}}$$

and carefully justify the coefficients.

Solution. (a) $U'(C) = V_W$. (b) FOC w.r.t. ω:

$$V_W W(\mu - r\mathbf{1}) + V_{WW}W^2\Sigma\omega + V_{Wz}W\sigma_z = 0,$$

so

$$\Sigma\omega = -\frac{V_W}{V_{WW}W}(\mu - r\mathbf{1}) - \frac{V_{Wz}}{V_{WW}W}\sigma_z, \qquad \omega = -\frac{V_W}{V_{WW}W}\Sigma^{-1}(\mu - r\mathbf{1}) - \frac{V_{Wz}}{V_{WW}W}\Sigma^{-1}\sigma_z.$$

(c) With CRRA, $\gamma = -\frac{WV_{WW}}{V_W}$ so $-\frac{V_W}{V_{WW}W} = \frac{1}{\gamma}$ and $-\dfrac{V_{Wz}}{V_{WW}W} = \dfrac{V_{Wz}}{\gamma V_W}$. Hence

$$\omega^* = \frac{1}{\gamma}\Sigma^{-1}(\mu - r\mathbf{1}) + \frac{V_{Wz}}{\gamma V_W}\Sigma^{-1}\sigma_z.$$

8. The z-mimicking portfolio and the ICAPM

Define the *z-mimicking* (hedging) portfolio q by

$$\Sigma q = \sigma_z, \qquad r_z \equiv q^\top r.$$

(a) Show that $\mathrm{Cov}(r_i, r_z) = e_i^\top \Sigma q = \sigma_{iz}$ and $\mathrm{Var}(r_z) = q^\top\Sigma q = \sigma_z^\top\Sigma^{-1}\sigma_z$.

(b) Let δ be the market portfolio. Argue (e.g., by projection of $\mu - r\mathbf{1}$ onto the span of $\Sigma\delta$ and σ_z) that there exist scalars λ_m, λ_z such that

$$\mu - r\mathbf{1} = \lambda_m\,\Sigma\delta + \lambda_z\,\sigma_z.$$

(c) Prove that

$$\lambda_m = \frac{\mu_m - r}{\mathrm{Var}(r_m)}, \qquad \lambda_z = \frac{\mu_z - r}{\mathrm{Var}(r_z)},$$

where $\mu_m \equiv \delta^\top\mu$ and $\mu_z \equiv q^\top\mu$.

(d) Taking the ith component, derive the **ICAPM**:

$$\boxed{\mu_i - r = \beta_{im}(\mu_m - r) + \beta_{iz}(\mu_z - r)}, \quad \beta_{im} \equiv \frac{\mathrm{Cov}(r_i, r_m)}{\mathrm{Var}(r_m)}, \quad \beta_{iz} \equiv \frac{\mathrm{Cov}(r_i, r_z)}{\mathrm{Var}(r_z)}.$$

Solution. (a) By definition of q, $\mathrm{Cov}(r_i, r_z) = e_i^\top \Sigma q = e_i^\top \sigma_z = \sigma_{iz}$ and $\mathrm{Var}(r_z) = q^\top \Sigma q = \sigma_z^\top \Sigma^{-1} \sigma_z$. (b) Since optimal policies span $\Sigma\delta$ (market risk) and σ_z (state risk), the pricing kernel implies $\mu - r\mathbf{1}$ lies in their span. Thus $\mu - r\mathbf{1} = \lambda_m \Sigma\delta + \lambda_z \sigma_z$. (c) Premultiply by $\delta^\top$: $\mu_m - r = \lambda_m\, \delta^\top \Sigma \delta = \lambda_m \mathrm{Var}(r_m)$, hence $\lambda_m = (\mu_m - r)/\mathrm{Var}(r_m)$. Premultiply by $q^\top$: $\mu_z - r = \lambda_z\, q^\top \sigma_z = \lambda_z \mathrm{Var}(r_z)$ since $q^\top \sigma_z = q^\top \Sigma q = \mathrm{Var}(r_z)$. (d) Take the ith component:

$$\mu_i - r = \lambda_m\, e_i^\top \Sigma \delta + \lambda_z\, e_i^\top \sigma_z = \frac{\mu_m - r}{\mathrm{Var}(r_m)} \mathrm{Cov}(r_i, r_m) + \frac{\mu_z - r}{\mathrm{Var}(r_z)} \mathrm{Cov}(r_i, r_z),$$

which yields the stated ICAPM with β_{im}, β_{iz}.

9. Optional check: Myopic vs. hedging demand

Briefly interpret the two components of ω^* in Exercise 7(c) and explain when the hedging term vanishes.

Solution. The myopic demand $\frac{1}{\gamma}\Sigma^{-1}(\mu - r\mathbf{1})$ trades off instantaneous mean and variance given current opportunity set. The hedging demand $\frac{V_{Wz}}{\gamma V_W}\Sigma^{-1}\sigma_z$ offsets unfavorable shifts in future investment opportunities; it vanishes if $V_{Wz} = 0$ (e.g., no priced z-risk or preferences/opportunity set make z irrelevant).

References

- Merton, Robert C. (1972). *An Analytic Derivation of the Efficient Portfolio Frontier. Journal of Financial and Quantitative Analysis*, 7(4), 1851–1872.
- Merton, Robert C. (1973). *An Intertemporal Capital Asset Pricing Model. Econometrica*, 41(5), 867–887.