Consumption Based Asset Pricing

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Introduction

These slides are based on Chapter 6 in Campbell (2018).

- The Equity Premium Puzzle (why is the average return on stocks so high)
- The Riskfree rate puzzle (why is the riskless rate so low and stable)
- The equity volatility puzzle (why stocks are more volatile than aggregate consumption growth)
- One resolution, habit formation

Lognormal Consumption with Power Utility

The log version of the fundamental equation of asset pricing assuming M and R are jointly lognormal. if X is lognormal

$$\log \mathbf{E}[X] = \mathbf{E}[\log(X)] + \frac{1}{2} Var \log(X) \tag{1}$$

using the fundamental asset pricing equation

$$\begin{aligned}
1 &= \mathbf{E}[M_{t+1} R_{t+1}] \\
0 &= \mathbf{E} \log(M_{t+1} R_{t+1}) + \frac{1}{2} Var \log(M_{t+1} R_{t+1}) \\
0 &= \mathbf{E} m_{t+1} + \mathbf{E} r_{t+1} + \frac{1}{2} Var(m_{t+1} + r_{t+1}) \\
0 &= \mathbf{E} m_{t+1} + \mathbf{E} r_{t+1} + \frac{1}{2} Var(m_{t+1}) + \frac{1}{2} Var(r_{t+1}) + Cov(m_{t+1}, r_{t+1}) \\
0 &= \mathbf{E} m_{t+1} + \mathbf{E} r_{t+1} + \frac{\sigma_m^2}{2} + \frac{\sigma_r^2}{2} + \sigma_{rm} \\
r_{f,t+1} &= -\mathbf{E} m_{t+1} - \frac{\sigma_m^2}{2}
\end{aligned} \tag{2}$$

Lognormal Consumption with Power Utility

Consider a representative agent with time-separable power utility with time discount factor δ and constant relative risk aversion γ defined over aggregate consumption C_t. Assume aggregate consumption has a conditional log-normal distribution (consumption growth is normal distributed)

$$U(C_t) = \frac{C_t^{1-\gamma} - 1}{1-\gamma} \tag{3}$$

these preferences imply that the SDF is

$$M_{t+1} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \tag{4}$$

or in logs

$$m_{t+1} = \log(\delta) - \gamma \Delta c_{t+1} \tag{5}$$

Lognormal Consumption with Power Utility

We now assume that consumption and asset returns are jointly conditionally homoskedastic. For any asset i

$$0 = \mathbf{E}r_{i,t+1} + \log\delta - \gamma\mathbf{E}\Delta c_{t+1} + \frac{1}{2}\left(\sigma_i^2 + \gamma^2\sigma_c^2 - 2\gamma\sigma_{ic}\right) \tag{6}$$

and for the risk free rate

$$r_{f,t+1} = -\log\delta + \gamma\mathbf{E}_t\Delta c_{t+1} - \frac{\gamma^2\sigma_c^2}{2} \tag{7}$$

It is linear in expected consumption growth, with slope coefficient equal to the coefficient of relative risk aversion. Moreover

$$\mathbf{E}[r_{i,t+1} - r_{f,t+1}] + \frac{\sigma_i^2}{2} = \gamma\sigma_{ic} \tag{8}$$

Three Puzzles

We now summarize empirical evidence in Campbell (2003), Hansen and Singleton (1983), and Mehra and Prescott (1985).

Table 6.1. *International Stock and Bill Returns and Consumption Growth*

Country	Sample period	$\overline{r_e}$	$\sigma(r_e)$	$\rho(r_e)$	$\overline{r_f}$	$\sigma(r_f)$	$\rho(r_f)$	$\overline{\Delta c}$	$\sigma(\Delta c)$	$\rho(\Delta c)$
USA	1947Q2–2011Q2	6.85	15.98	0.10	0.72	1.78	0.36	1.74	1.64	0.04
Australia	1970Q1–2011Q2	3.84	20.75	0.03	2.00	2.18	0.55	1.82	1.77	–0.09
Canada	1970Q1–2011Q2	5.47	17.85	0.15	2.07	1.64	0.66	1.69	1.93	0.07
France	1973Q2–2011Q2	7.06	23.10	0.08	2.08	1.53	0.74	1.38	1.80	–0.13
Germany	1978Q4–2011Q2	7.54	23.85	0.04	2.38	1.09	0.40	1.74	4.19	–0.07
Italy	1971Q2–2011Q2	1.51	25.74	0.07	1.86	2.06	0.77	2.18	2.23	0.47
Japan	1970Q2–2011Q2	2.70	21.41	0.09	1.03	1.91	0.29	1.72	2.92	–0.10
Netherlands	1977Q2–2011Q2	8.57	19.76	0.09	2.29	1.42	0.45	1.05	2.21	–0.11
Sweden	1970Q1–2011Q2	8.93	25.16	0.12	1.68	2.23	0.40	1.23	1.81	–0.15
Switzerland	1982Q2–2011Q2	8.14	20.05	0.01	0.87	1.32	0.03	0.75	1.30	–0.22
UK	1970Q1–2011Q2	6.33	19.85	0.10	1.34	2.60	0.46	2.14	2.68	–0.03

Three Puzzles

(Campbell 2018, Page 163) The basic message of the table is that stock returns are high on average, highly volatile, and only weakly autocorrelated; ex post real interest rates are much lower on average, modestly volatile, and positively autocorrelated; and real consumption growth is also low and stable on average with autocorrelations that vary considerably across countries.

The Equity Premium Puzzle

What should be γ in order to fit the above equations with real data? Reasonable numbers should be between 1 and 10.

Table 6.2. The Equity Premium Puzzle

Country	Sample period	$\overline{er_e}$	$\sigma(er_e)$	$\sigma(m)$	$\sigma(\Delta c)$	$\rho(er_e, \Delta c)$	RRA(1)	RRA(2)
USA	1947Q2–2011Q2	7.39	15.86	46.57	1.64	0.18	154.98	28.42
Australia	1970Q1–2011Q2	3.95	20.58	19.21	1.77	–0.11	<0	10.89
Canada	1970Q1–2011Q2	5.01	17.93	27.94	1.93	0.09	166.97	14.51
France	1973Q2–2011Q2	7.68	23.22	33.06	1.80	0.00	<0	18.34
Germany	1978Q4–2011Q2	8.03	23.94	33.54	4.19	–0.01	<0	8.01
Italy	1971Q2–2011Q2	2.96	25.71	11.51	2.23	0.08	66.96	5.15
Japan	1970Q2–2011Q2	3.95	21.33	18.49	2.92	0.05	118.09	6.32
Netherlands	1977Q2–2011Q2	8.22	19.70	41.72	2.21	0.13	141.29	18.90
Sweden	1970Q1–2011Q2	10.44	25.28	41.32	1.81	0.07	314.53	22.87
Switzerland	1982Q2–2011Q2	9.27	20.01	46.33	1.30	0.07	483.74	35.60
UK	1970Q1–2011Q2	6.99	19.96	35.00	2.68	–0.04	<0	13.07

The Riskfree Rate Puzzle

- One response to the equity premium puzzle is simply to accept that investors are more risk averse than commonly believed. However, high risk aversion has implausible implications for the riskless real interest rate in equilibrium, a fact described as the riskfree rate puzzle by Weil (1989).
- The rate of time preference must typically be negative to fit the mean real interest rate with risk aversion coefficients

The Riskfree Rate Puzzle

Table 6.3. The Riskfree Rate Puzzle

Country	Sample period	$\overline{r_f}$	$\overline{\Delta c}$	$\sigma(\Delta c)$	RRA(1)	TPR(1)	RRA (2)	TPR(2)
USA	1947Q2–2011Q2	0.72	1.74	1.64	154.98	53.08	28.42	–37.97
Australia	1970Q1–2011Q2	2.00	1.82	1.77	<0	N/A	10.89	–15.93
Canada	1970Q1–2011Q2	2.07	1.69	1.93	166.97	236.67	14.51	–18.55
France	1973Q2–2011Q2	2.08	1.38	1.80	<0	N/A	18.34	–17.80
Germany	1978Q4–2011Q2	2.38	1.74	4.19	<0	N/A	8.01	–5.97
Italy	1971Q2–2011Q2	1.86	2.18	2.23	66.96	–32.42	5.15	–8.73
Japan	1970Q2–2011Q2	1.03	1.72	2.92	118.09	394.17	6.32	–8.15
Netherlands	1977Q2–2011Q2	2.29	1.05	2.21	141.29	340.11	18.90	–8.86
Sweden	1970Q1–2011Q2	1.68	1.23	1.81	314.53	1230.28	22.87	–17.84
Switzerland	1982Q2–2011Q2	0.87	0.75	1.30	483.74	1617.47	35.60	–15.24
UK	1970Q1–2011Q2	1.34	2.14	2.68	<0	N/A	13.07	–20.48

The Equity Volatility Puzzle

- This is a puzzle if stocks are a claim to aggregate consumption (i.e., if stock dividends equal consumption), consumption growth is iid, and investors have power utility, because then the consumption-wealth ratio should be constant, and stock returns and consumption growth should have the same volatility.
- Campbell (2003) calls this the equity volatility puzzle and argues that it is just as fundamental a challenge for consumption-based asset pricing models as either of the other two puzzles.
- Volatility can be explained if dividends or consumption growth have predictable long-run components, as in the long-run risks model of Bansal and Yaron (2004), or if preferences induce persistent fluctuations in risk premia, as in the Campbell and Cochrane (1999) model of habit formation

Responses to the Puzzles

- **Sampling error**. The risk aversion coefficients do not come with standard errors. In some cases, a confidence interval includes more modest values that are easier to reconcile with intuition.
- **Sample selection**. We may be looking at an unusual sample drawn from the right tail of the ex ante distribution of stock returns. Dimson, Marsh, and Staunton (2002) have argued that the late 20th century was a particularly fortunate period for economic growth and investors in long-term assets. Fama and French (2002) and McGrattan and Prescott (2005) have argued that this period saw a decline in the required real stock return (attributed by McGrattan and Prescott to declining capital income taxation) that drove up stock prices in a manner that was not expected ex ante.
- **Mismeasurement of returns**. McGrattan and Prescott (2003) emphasize that capital income taxation reduces average after-tax returns to taxable investors, although it also reduces the volatility of these returns.

Responses to the Puzzles

- **Mismeasurement of consumption**. Measured consumption is a flow over a period of time, whereas the theoretical concept required to test a consumption-based asset pricing model is instantaneous consumption at a point in time, which is then used to construct consumption growth over a discrete interval (Breeden, Gibbons, and Litzenberger 1989). In addition, the usual National Income and Product Accounts (NIPA) consumption series have been filtered to reduce measurement error and seasonally adjusted, and this may distort the covariance of stock returns with consumption. Savov (2011) works with the volume of garbage, a side-effect of consumption that is measured without filtering or seasonal adjustment, and obtains lower although still substantial estimates of risk aversion; Kroencke (2017) obtains similar results by unfiltering NIPA consumption.
- **Long-run consumption covariances**. Perhaps stock market risk should be measured by the covariance between stock returns and long-run consumption growth, not shortrun consumption growth (Daniel and Marshall 1997, Parker and Julliard 2005). This might be because measurement issues are less serious at long horizons, or because there are costs of processing information that lead to infrequent adjustment of consumption (Gabaix and Laibson 2001).

Responses to the Puzzles

- **Rare disasters**. Consumption growth is not lognormal but has a fat-tailed distribution. Standard preferences imply that expected returns are higher than for a lognormal consumption distribution with the same variance. The fat tail in consumption growth may arise from a small probability of a disaster (Rietz 1988, Barro 2006) or from parameter uncertainty (Weitzman 2007).
- **Epstein-Zin preferences**. It may be that the power utility model does not adequately represent preferences. A popular alternative is Epstein-Zin utility (Epstein and Zin 1989, 1991), which retains the basic scale independence of power utility but abandons the restriction that the elasticity of intertemporal substitution is the reciprocal of the coefficient of relative risk aversion. At the very least, this allows one to increase risk aversion (to fit the equity premium) without encountering the riskfree rate puzzle.
- **Ambiguity aversion**. The engineering literature on robust optimal control proposes to handle model uncertainty by optimizing in a manner that delivers satisfactory results even if an unfavorable model is true. In economics, this approach can be used as a model of optimizing behavior by investors who are uncertain about the true model.

Responses to the Puzzles

- **Long-run risks**. Even if consumption is adjusted continuously, investors are averse to covariance with long-run consumption growth if they have Epstein-Zin preferences with a relatively high elasticity of intertemporal substitution (Restoy and Weil 2011, Bansal and Yaron 2004, Bansal, Kiku, and Yaron 2012). Models with such preferences and persistent variation in the consumption growth rate and the volatility of consumption have come to be known as long-run risk models.
- **Habit formation**. Models in which investors get utility from consumption relative to a gradually adjusting habit level can generate gradual adjustment in consumption (Constantinides 1990) and time varying risk aversion (Campbell and Cochrane 1999). Time-varying risk aversion contributes to an explanation of the equity volatility puzzle.
- **Nonseparable utility**. The standard assumption in the literature is that marginal utility derived from consumption of nondurables and services (the consumption series used in most empirical tests) does not depend on the consumption of leisure or of services provided by a stock of durable goods. If this assumption fails- that is, if utility is nonseparable across nondurables and services, on the one hand, and leisure or durable goods, on the other -then standard tests are misspecified (Dunn and Singleton 1986, Eichenbaum and Hansen 1990, Yogo 2006)

Responses to the Puzzles

- **Uninsurable idiosyncratic risk**. Constantinides and Duffie (1996) show that in principle, with an arbitrary distribution of uninsurable background risk, the equity premium can be arbitrarily large with an arbitrary coefficient of risk aversion for investors.
- **Portfolio restrictions.** The effects of heterogeneity can be amplified if there are restrictions on investor portfolios, for example, participation constraints that prevent some consumers from holding stocks, borrowing constraints that limit the risk exposures of some investors, or barriers to capital flows across countries

Habit Formation

- We now consider models of habit formation, in which the marginal utility of consumption depends not on the absolute level of consumption, but on consumption relative to a stochastic ”habit” process that is related to the history of consumption.
- We write the period utility function as $u(C_t, X_t)$ where X_t is the time-varying habit or subsistence level.
- Results differ if the functional form of $u()$ includes a ratio C_t/X_T or a difference $C_t - X_t$.

A Ratio Model of Habit

Abel (1990)

$$U_t = \sum_{s=0}^{\infty} \delta^s \frac{(C_{t+s}/X_{t+s})^{1-\gamma} - 1}{1-\gamma} \tag{9}$$

and a Consumption Habit

$$X_t = C_{t-1}^{\kappa} \tag{10}$$

This implies

$$M_{t+1} = \delta \left(\frac{C_t}{C_{t-1}} \right)^{\kappa(\gamma-1)} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \tag{11}$$

A Ratio Model of Habit

Assuming homoskedasticity and joint lognormality of asset returns and consumption growth, this implies the following riskless real interest rate:

$$r_{f,t+1} = -\log\delta - \gamma^2\frac{\sigma_c^2}{2} + \gamma\mathbf{E}_t\Delta c_{t+1} - \kappa(\gamma-1)\Delta c_t \quad (12)$$

This model of habit alters the riskless real rate, but it does not change the risk premium since the innovarion in the log SDF is the same. (The extra element of the SDF is known as time t)

The Campbell-Cochrane Model

Campbell and Cochrane 1999

- $\Delta c_{t+1} = g + \epsilon_{c,t+1}$
- $Var(\epsilon_{c,t+1} = \sigma_c^2$

$$U_t = \mathbf{E}_t = \sum_{s=1}^{\infty} \delta^s \frac{(C_{t+s} - X_{t+s})^{1-\gamma} - 1}{1-\gamma} \tag{13}$$

It is convenient to capture the relation between consumption and habit by the surplus consumption ratio St , defined by

$$S_t = \frac{C_t - X_t}{C_t} \tag{14}$$

if habit is fixed the local coefficient of relative risk aversion is

$$-CU_{CC}/U_c = \gamma/S_t \tag{15}$$

Dynamics of the Habit Model

Assume habit and consumption move stochastically

$$s_{t+1} = (1-\phi)\bar{s} + \phi s_t + \underbrace{\lambda(s_t)}_{\text{Sensitivity Function}} \epsilon_{c,t+1} \tag{16}$$

since $s_t = \log(1 - \exp(x_t - c_t))$

$$x_{t+1} \approx (1-\phi)a + \phi x_t + (1-\phi)c_t \tag{17}$$

Dynamics of the Habit Model

$$
\begin{aligned}
u'(C_t) &= (C_t - X_t)^{-\gamma} = S_t^{-\gamma} C_t^{-\gamma} \\
M_{t+1} &= \left(\frac{S_{t+1}}{S_t}\right)^{-\gamma} \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \\
r_{f,t+1} &= -\log\delta + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) - \frac{\gamma^2 \sigma_c^2}{2}(1 + \lambda(s_t))^2
\end{aligned}
\tag{18}
$$

Assuming a functional form for $\lambda(s_t)$ CC calibrate their model to fit postwar quarterly US data, choosing the mean consumption growth rate $g = 1.89\%$ and $\sigma_c = 1.50\%$ per year. Their model implies a reasonable risk aversion parameter of $\gamma \approx 2$.

The Joint Hypothesis

- What does it mean that markets are efficient? They ”fully” reflect available information.
- What does fully mean? We need to make a normative statement and specify the process of price formation.

$$\begin{aligned} \mathbb{E}[1 + R_{i,t+1}] = (1 + R_{f,t+1})(1 - Cov_t(M_{t+1}, R_{i,t+1})) = Z_{it} \\ 1 + R_{i,t+1} = Z_{it} + u_{i,t+1} \end{aligned} \quad (19)$$

- We can test if $u_{i,t+1}$ is unpredictable only if we correctly specified $Cov_t(M_{t+1}, R_{i,t+1})$.

The Generalized Method of Moments (Fast Course)

- Hansen (and others) exploit the joint hypothesis realizing that unpredictable abnormal returns can be used to estimate model parameters.

Consider a parametrization $M_{t+1}(\theta)$, e.g. $M_{t+1} = a + bR_{m,t+1}$

$$\mathbb{E}[M_{t+1}(\theta)(1 + R_{t+1}) - 1] = \mathbb{E}[u_{t+1}(\theta)] \tag{20}$$

The idea is to find parameters θ that make linear combinations of the sample counterpart

$$g_T(\theta) = \frac{1}{T}\sum_{t=1}^{T} u_t(\theta) \tag{21}$$

equal to zero using a quadratic form

$$\min g_T' W g_T \tag{22}$$

for some weighting matrix W. The efficiency and finite sample properties of $\hat{\theta}$ will depend on the choice of W and a sequence of re-estimations.

The GMM Estimator

Hansen (1982) Large Sample Properties of Generalized Method of Moments, Econometrica. Two-stage procedure, for any positive semidefined matrix W, e.g. I

$$\begin{aligned}\hat{\theta}_1 &= \arg\min_{\theta} g_T(\theta)' W g_T(\theta) \\ \frac{\partial g_T(\theta)}{\partial \theta} W g_T(\theta) &= a g_T(\theta) = 0\end{aligned} \tag{23}$$

This estimator is consistent and asymptotically normal but not always efficient, the efficient estimator is obtained by estimating S as the covariance of moments u_t, re-estimate

$$\hat{\theta}_2 = \arg\min_{\theta} g_T(\theta)' \hat{S}^{-1} g_T(\theta) \tag{24}$$

Asymptotics, define $a = \frac{\partial g_T(\theta)'}{\partial \theta} \hat{S}^{-1}$, $d = \frac{\partial g_T(\theta)}{\partial \theta'}$

$$\sqrt{T}(\theta - \hat{\theta}) \rightarrow \mathcal{N}\left[0, (d' S^{-1} d)^{-1}\right] \tag{25}$$

The GMM Estimator

How good is the estimator? (How close to zero are the moment conditions?)

$$T \underbrace{g_T(\hat{\theta})' \hat{S}^{-1} g_T(\hat{\theta})}_{\text{Hansen's } J} \sim \chi^2_{\text{moments-parameters}} \tag{26}$$

GMM in Practice

Consider the consumption based model with power utility. How would you estimate γ without assuming lognormality in consumption?

$$1 = \mathbb{E}[M_{t+1}R_{t+1}|I_t] \tag{27}$$

in practice you can use instruments z_t (variables belonging to the information set)

$$\mathbb{E}[(M_{t+1}R_{t+1} - 1)z_t] = 0 \tag{28}$$

consider the simplest case where $z_t = 1$, and J testing assets

$$\mathbb{E}[(\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}R_{t+1} - 1] = 0 \tag{29}$$

The GMM Recipe

Compute the sample moments

$$g_T(\gamma) = \frac{1}{T}\sum_{t=0}^{T}(\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} R_{t+1} - 1) \quad (30)$$

solve

$$\arg\min g_T(\hat{\gamma})' W g_T(\hat{\gamma}) \quad (31)$$

for some p.s.d. matrix W. Normally I for robustness, $cov(g_T)^{-1}$ for efficiency.
Compute standard errors given the asymptotics of $\hat{\gamma}$.