Optimization using Solver

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Optimization

Notation and Definitions

- **Optimization**: The process of finding the minimum (or maximum) value of a function subject to a set of constraints.
- **Objective function**: The function to be minimized or maximized.
- **Decision variables**: The variables that can be adjusted to optimize the objective function.
- **Constraints**: The limitations on the decision variables.

Easy Optimization: Convex Optimization

Convex Optimization

- **Convex function**: A function that has a non-negative second derivative.
- **Convex set**: A set where the line segment between any two points in the set lies entirely within the set.
- Minimization of a convex function over a convex set is a convex optimization problem. These problems are easy to solve because they have a single global minimum.

Example: Linear Programming

Linear Programming

- **Objective function**: A linear function.
- **Constraints**: Linear inequalities.
- **Decision variables**: Continuous.
- **Optimization**: Minimize or maximize the objective function.

Linear Programming Problem: Investment Portfolio Optimization

You are managing an investment portfolio and have two options for investment:

- **Investment A**: This is a low-risk investment with a 5% return.
- **Investment B**: This is a higher-risk investment with a 10% return.

Your goal is to maximize the return on your investment, but there are constraints:

- The total amount available to invest is $100,000.
- You decide that no more than 60% of the total investment should be in the higher-risk investment (Investment B).
- Due to diversification, you want to invest at least $40,000 in Investment A.

Decision Variables:

Let:

- x_1 be the amount invested in Investment A.
- x_2 be the amount invested in Investment B.

Objective Function (Maximize total return):

$$\text{Maximize } Z = 0.05x_1 + 0.10x_2$$

Constraints:

1. Total amount invested:

$$x_1 + x_2 \leq 100,000$$

2. Limit on investment in higher-risk option:

$$x_2 \leq 0.60(x_1 + x_2)$$

(This simplifies to $x_2 \leq 60,000$.)

3. Minimum investment in Investment A:

$$x_1 \geq 40,000$$

4. Non-negativity constraints:

$$x_1 \geq 0, \quad x_2 \geq 0$$

Linear Programming Formulation:

$$\text{Maximize } Z = 0.05x_1 + 0.10x_2$$

Subject to:

$$x_1 + x_2 \leq 100,000$$
$$x_2 \leq 60,000$$
$$x_1 \geq 40,000$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

Using Excel's Solver

- Installation: Go to `File > Options > Add-ins`.
- IN `Manage:` select `Excel Add-ins` and click `Go`. Install `Solver Add-in`.

Using Excel's Solver

- Define the parameters of the model in cells.

Asset	Return	Minimum Investment	Maximum Investment
A	5%	40000	
B	10%		60000
		Total Investment	100000

Define the objective function and constraints as formulas in Excel.

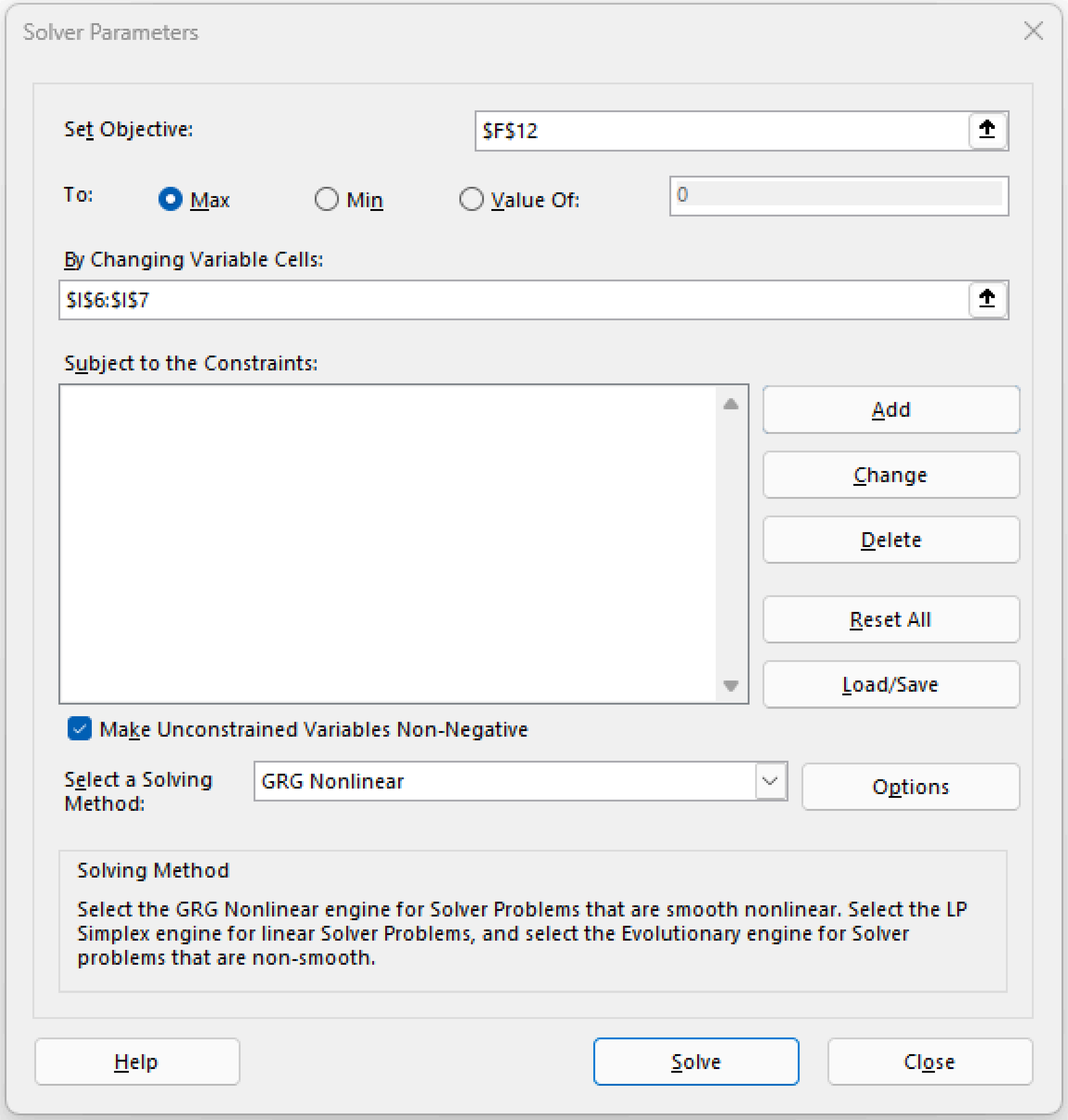
	E	F	G	H	I
5	**Asset**	**Return**	**Minimum Investment**	**Maximum Investment**	**Investment**
6	A	0.05	40000		
7	B	0.1		60000	
8					
9			**Total Investment**	100000	

12	**Obj**	=F6*I6+F7*I7
13	**C1**	=I6+I7
14	**C2**	=I6
15	**C3**	=I7

Solver: `Data > Solver`

	E	F	G	H	I
5	**Asset**	**Return**	**Minimum Investment**	**Maximum Investment**	**Investment**
6	A	5%	40000		
7	B	10%		60000	
8					
9			**Total Investment**	100000	

12	**Obj**	0
13	**C1**	0
14	**C2**	0
15	**C3**	0

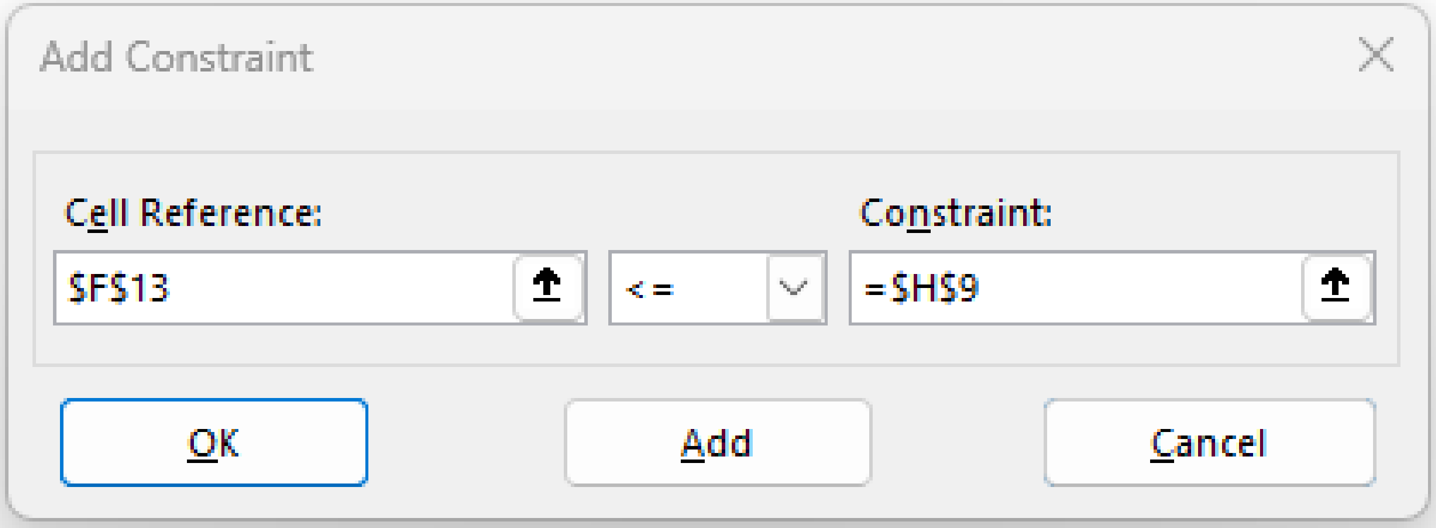


Constraints

Click `Add` to add the constraints.

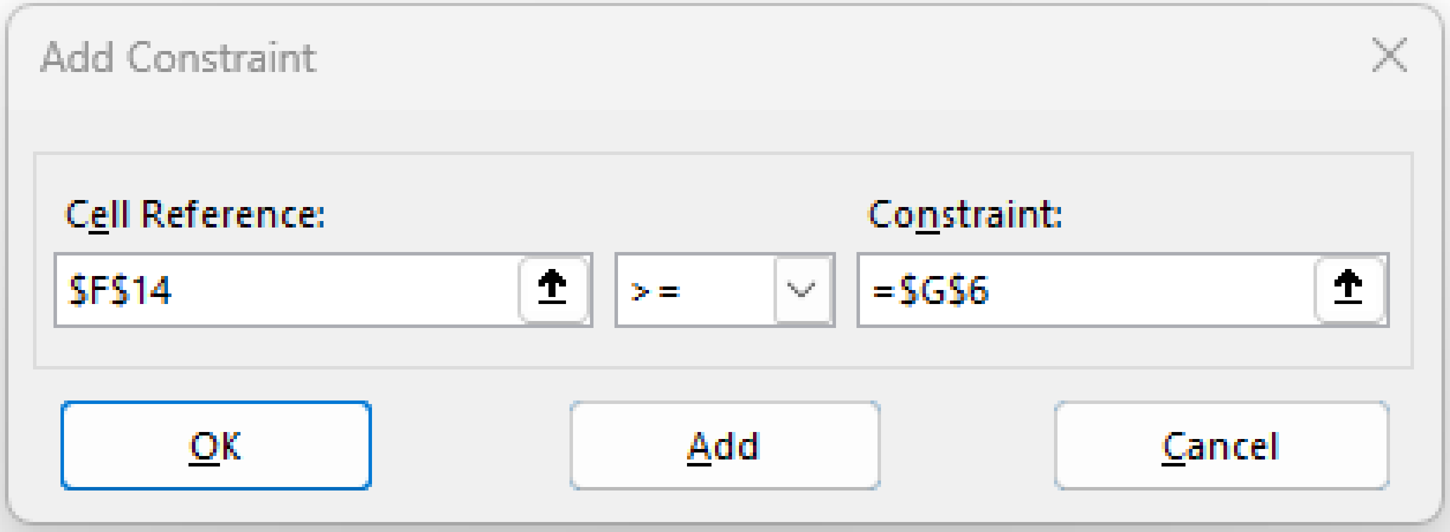
	E	F	G	H	I
5	**Asset**	**Return**	**Minimum Investment**	**Maximum Investment**	**Investment**
6	A	5%	40000		
7	B	10%		60000	
8					
9			**Total Investment**	100000	

12	**Obj**	0
13	**C1**	0
14	**C2**	0
15	**C3**	0



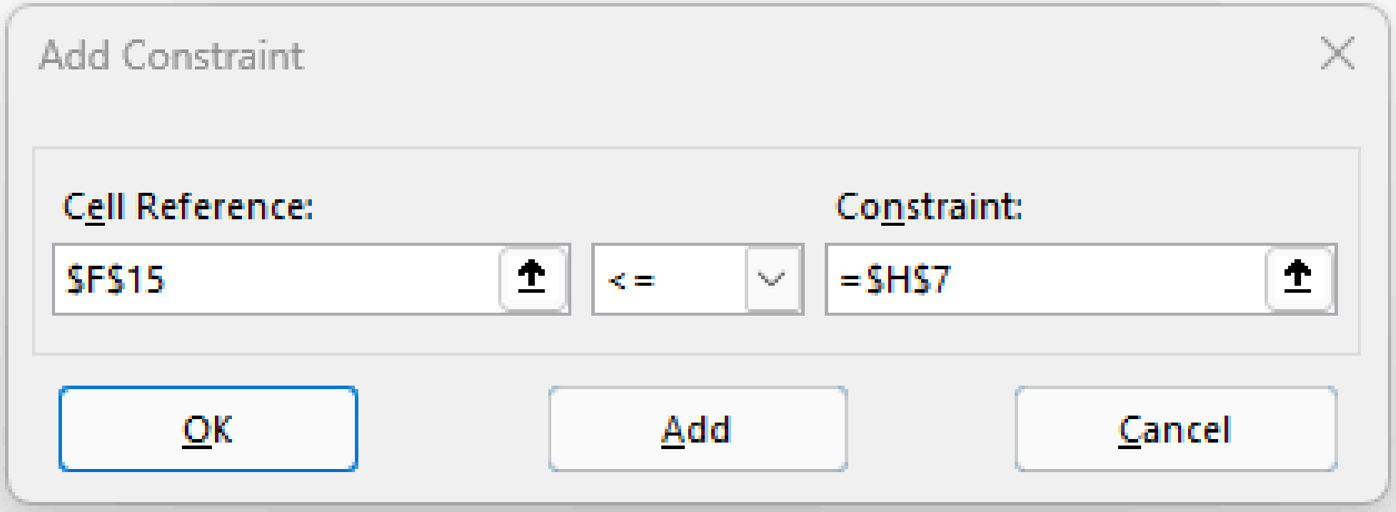
	E	F	G	H	I
5	**Asset**	**Return**	**Minimum Investment**	**Maximum Investment**	**Investment**
6	A	5%	40000		
7	B	10%		60000	
8					
9			**Total Investment**	100000	

12	**Obj**	0
13	**C1**	0
14	**C2**	0
15	**C3**	0



	E	F	G	H	I
5	**Asset**	**Return**	**Minimum Investment**	**Maximum Investment**	**Investment**
6	A	5%	40000		
7	B	10%		60000	
8					
9			**Total Investment**	100000	

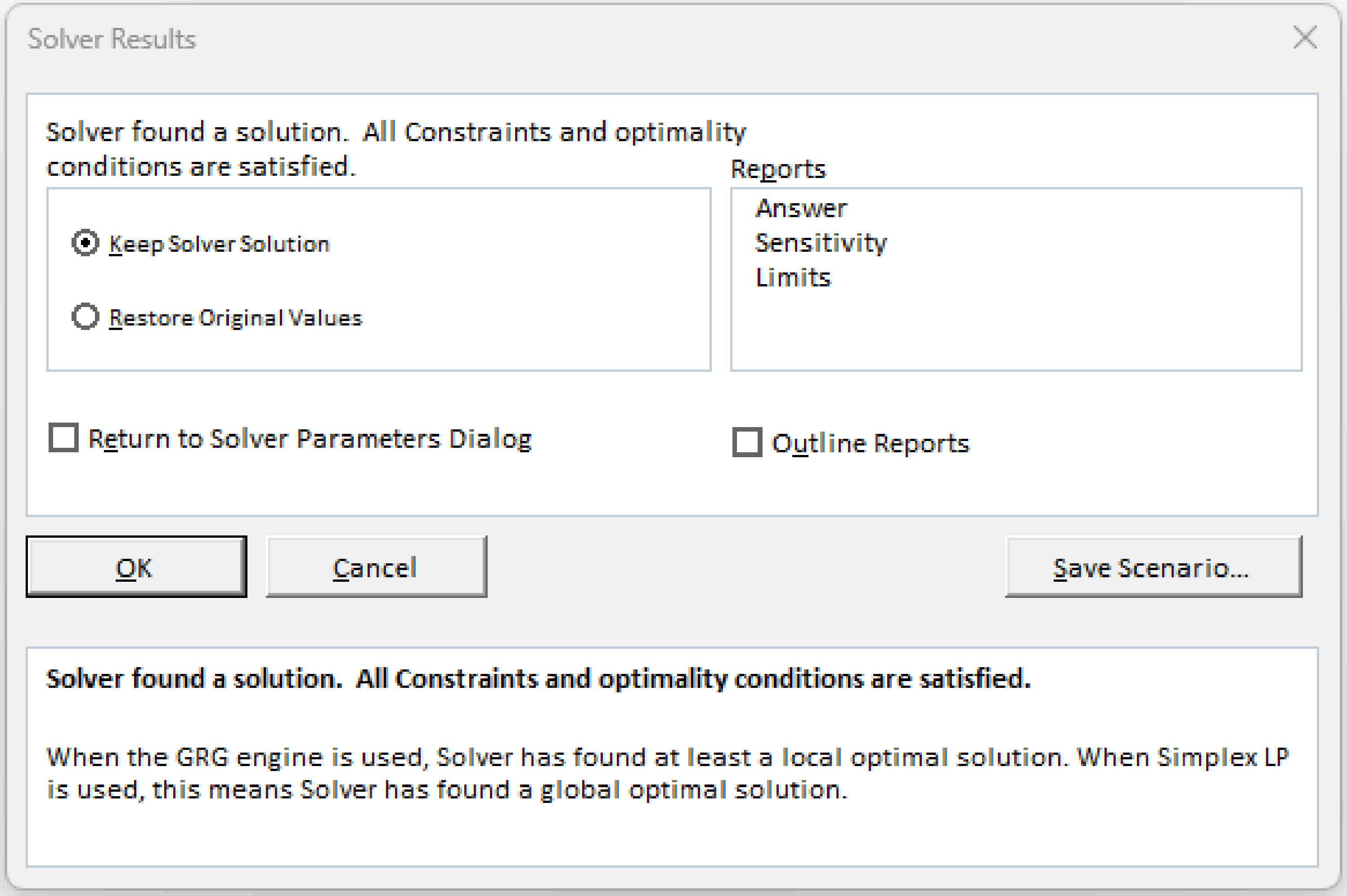
12	**Obj**	0
13	**C1**	0
14	**C2**	0
15	**C3**	0



- For bounds $x_1 \geq 0$ and $x_2 \geq 0$, we can use the `Non-Negative` option in Solver.
- The solving method can be set to `Simplex LP` (for linear programming problems).

	E	F	G	H	I
5	**Asset**	**Return**	**Minimum Investment**	**Maximum Investment**	**Investment**
6	A	5%	40000		40000
7	B	10%		60000	60000
8					
9			**Total Investment**	100000	

12	**Obj**	8000
13	**C1**	100000
14	**C2**	40000
15	**C3**	60000



Convex Optimization

- What if we also care about the risk of the investment?
- Volatility asset A: 2%
- Volatility asset B: 5%
- Correlation: 0.85
- Risk aversion: 1.0

The new objective function:

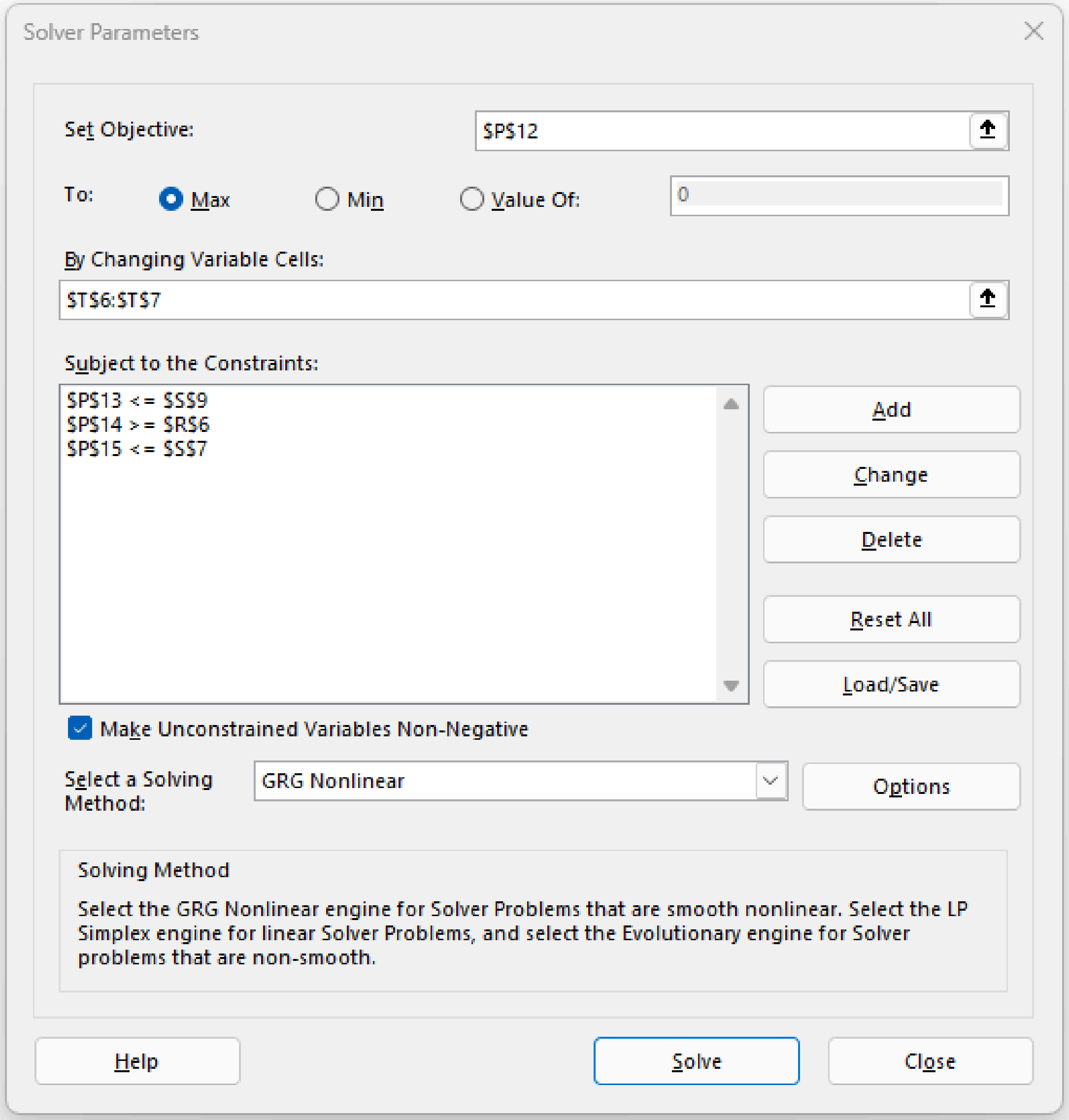
$$\text{Maximize } Z = 0.05x_1 + 0.10x_2 - \frac{1}{2} \times (0.02^2 \times x_1^2 + \frac{1}{2} \times 0.05^2 \times x_2^2 + 2 \times 0.02 \times 0.05 \times 0.85 \times x_1 \times x_2)$$

	O	P	Q	R	S	T
5	**Asset**	**Return**	**Volatility**	**Minimum Investment**	**Maximum Investment**	**Investment**
6	A	0.05	0.02	40000		40000
7	B	0.1	0.05		60000	60000
8		**Corr**	0.85			
9		**Gamma**	1	**Total Investment**	100000	

12	**Obj**	=P6*T6+P7*T7-(Q9/2)*((T6^2)*Q6^2+(T7^2)*Q7^2+2*T6*T7*Q6*Q7*Q8)
13	**C1**	=T6+T7
14	**C2**	=T6
15	**C3**	=T7

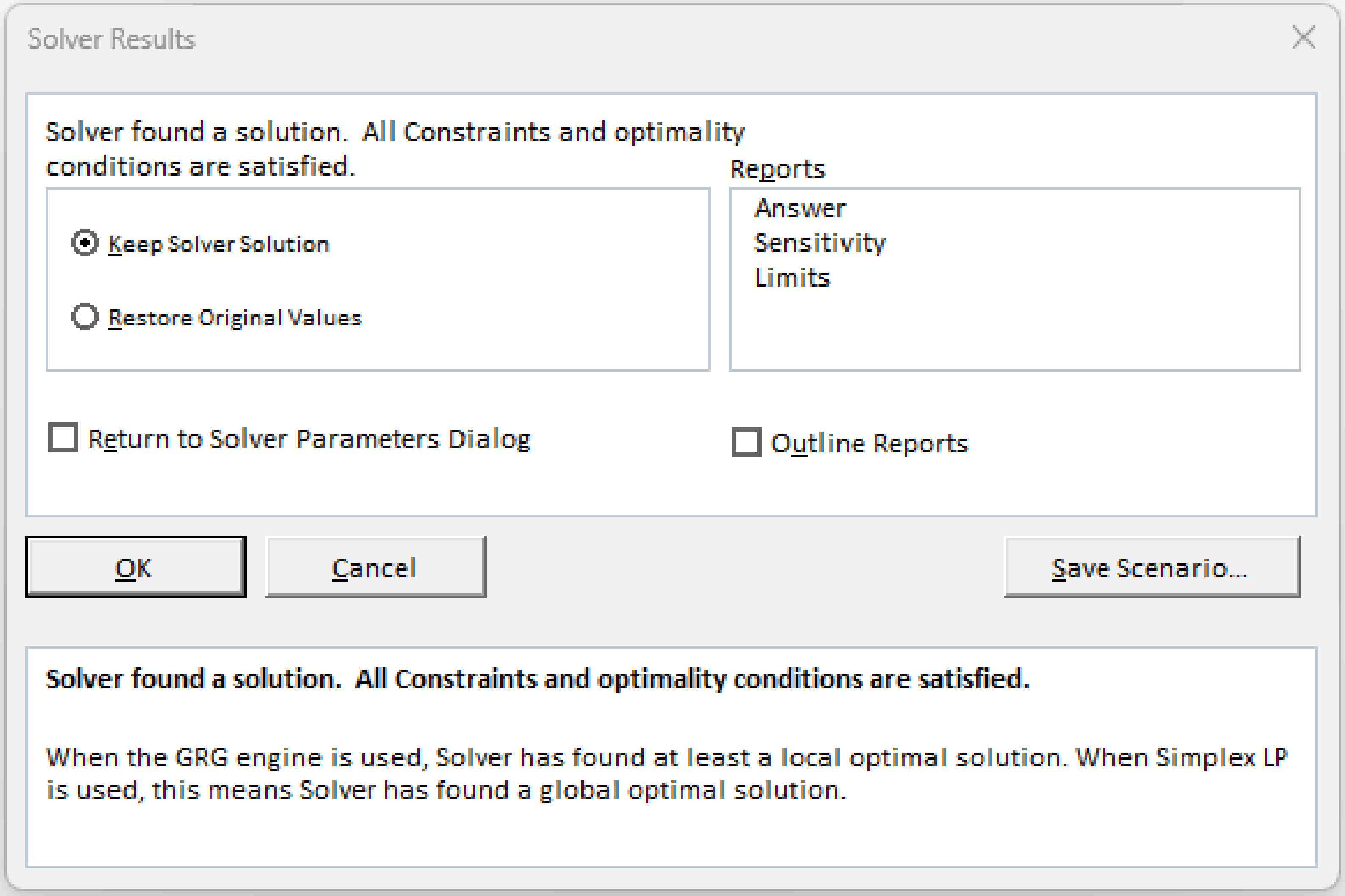
	O	P	Q	R	S	T
5	**Asset**	**Return**	**Volatility**	**Minimum Investment**	**Maximum Investment**	**Investment**
6	A	5%	2%	40000		40000
7	B	10%	5%		60000	60000
8		**Corr**	0.85			
9		**Gamma**	1	**Total Investment**	100000	

12	**Obj**	-6852000
13	**C1**	100000
14	**C2**	40000
15	**C3**	60000



5	Asset	Return	Volatility	Minimum Investment	Maximum Investment	Investment
6	A	5%	2%	40000		40000
7	B	10%	5%		60000	0
8		Corr	0.85			
9		Gamma	1	Total Investment	100000	

12	Obj	-318000
13	C1	40000
14	C2	40000
15	C3	0



MIP (Mixed-Integer Programming)

- Convex constraints are not enough to represent all real-world problems.
- Many problems require integer or binary decision variables. These variables can represent integer quantities or **binary** decisions that must be made during the optimization.

First Example: Buying Shares

- You can only buy shares in whole numbers. (Although you can pool your money with others to buy a fraction of a share.)
- In the last example, assume company A's shares cost $10 and company B's shares cost $20. How would you modify the model?

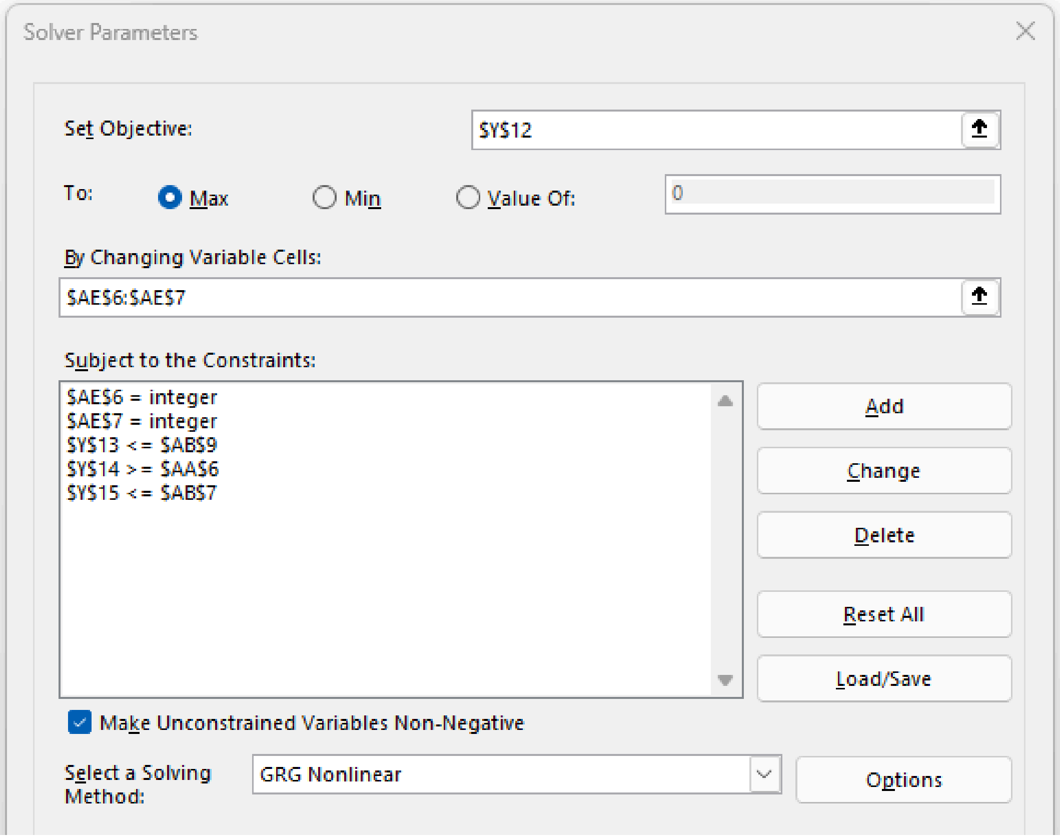
$$\text{Maximize } Z = 0.05 \times 10 \times n_1 + 0.10 \times 20 \times n_2$$

Subject to:

$$n_1 \times 10 + n_2 \times 20 \leq 100,000$$
$$n_2 \times 20 \leq 60,000$$
$$n_1 \times 10 \geq 40,000$$
$$n_1 \in \mathbb{N}_+, \quad n_2 \in \mathbb{N}_+$$

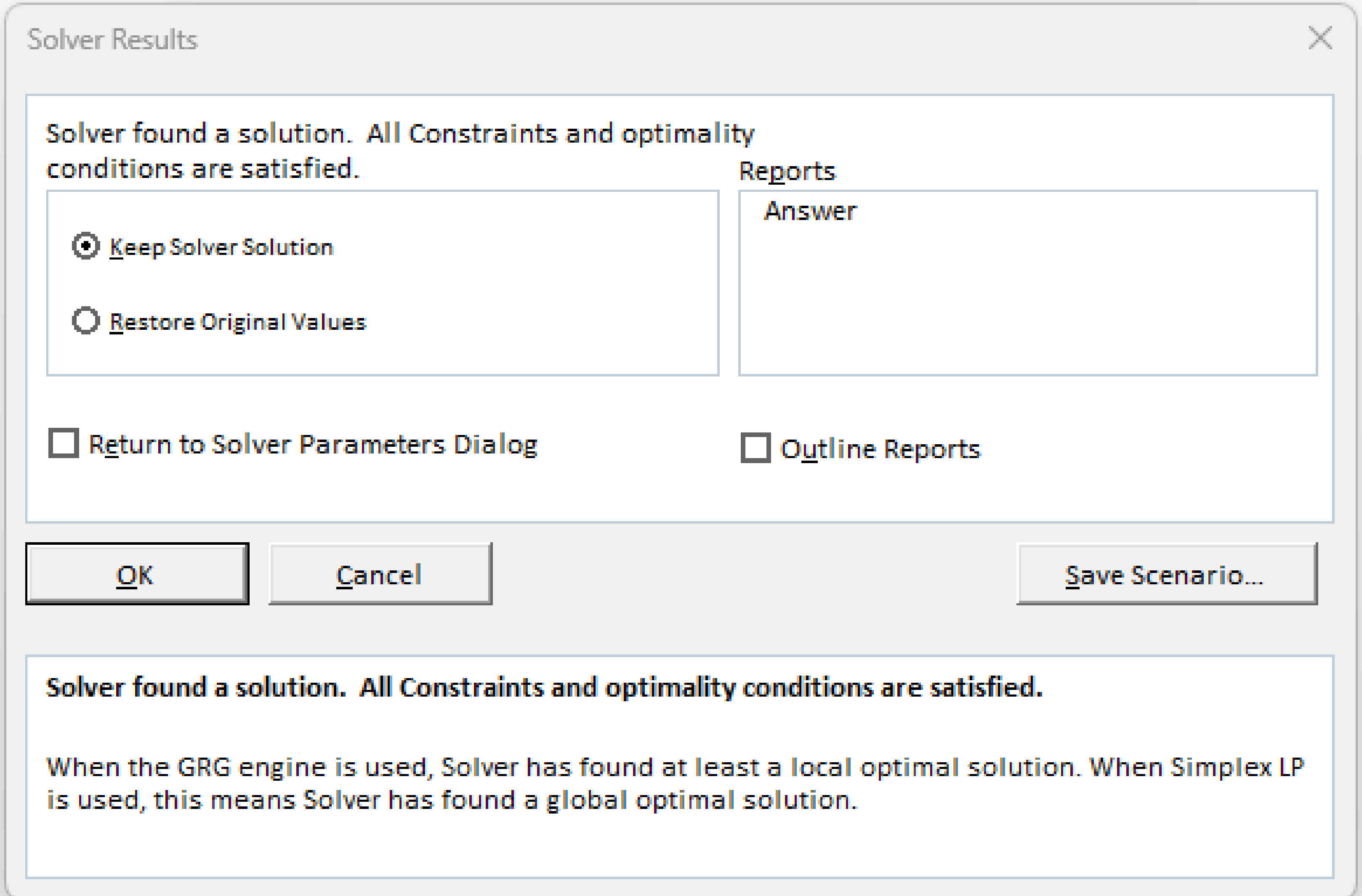
	O	P	Q	R	S	T		
5	**Asset**	**Return**	**Volatility**	**Minimum Investment**	**Maximum Investment**	**Investment**	**Price**	**Shares**
6	A	0.05	0.02	40000		=AD6*AE6	10	4000
7	B	0.1	0.05		60000	=AD7*AE7	20	0
8		**Corr**	0.85					
9		**Gamma**	1	**Total Investment**	100000			

12	**Obj**	=Y6*AC6+Y7*AC7-(Z9/2)*((AC6^2)*Z6^2+(AC7^2)*Z7^2+2*AC6*AC7*Z6*Z7*Z8)
13	**C1**	=AC6+AC7
14	**C2**	=AC6
15	**C3**	=AC7



	O	P	Q	R	S	T		
5	**Asset**	**Return**	**Volatility**	**Minimum Investment**	**Maximum Investment**	**Investment**	**Price**	**Shares**
6	A	5%	2%	40000		40000	10	4000
7	B	10%	5%		60000	0	20	0
8		**Corr**	0.85					
9		**Gamma**	1	**Total Investment**	100000			

12	**Obj**	-318000
13	**C1**	40000
14	**C2**	40000
15	**C3**	0



Second Example: Binary Decision

- Binary decisions are often used in optimization problems where a decision must be made between two options.
- Example: You are a manufacturer and have two options for a new product line: Product A and Product B. You can only choose one of them. How would you model this in an optimization problem?

The KnapSack Problem

- You have a knapsack with a weight limit of 15 kg.
- You have the following items to choose from: A, B, C, D, E, F.
- Each item has a weight and a utility value for you.

E.g., A: 5 kg, 10 value.

B: 3 kg, 7 value.

C: 4 kg, 8 value.

D: 6 kg, 12 value.

E: 2 kg, 6 value.

F: 7 kg, 14 value.

The KnapSack Problem

- You want to maximize the total value of the items you can carry in your knapsack without exceeding the weight limit.
- How would you model this as an optimization problem?
- Introduce binary decision variables y_i for each item i.

$$i \in \{A, B, C, D, E, F\} \quad y_i \in \{0, 1\}$$

weights and utility

$$w_i \quad \text{and} \quad v_i$$

Objective Function:

$$\text{Maximize } Z = \sum_{i \in \{A,B,C,D,E,F\}} v_i \times y_i$$

Subject to:

$$\sum_{i \in \{A,B,C,D,E,F\}} w_i \times y_i \leq 15$$

$$y_i \in \{0, 1\}$$

Seat Assignment Problem

- You are organizing a wedding and have to assign seats to attendees. Some attendees have requested to sit next to each other, while others have requested to sit apart (restrictions).
- You think the wedding will be more enjoyable if people with similar interests sit together (preferences).

Seat Assignment Problem

Attendees = $i = 1, \ldots, N$

Restrictions

$$r_{ij} = \begin{cases} 1 & \text{if attendees } i \text{ and } j \text{ must sit together} \\ 0 & \text{otherwise} \end{cases}$$

Preferences p_{ij}, the higher the value, the more fun the wedding will be if attendees i and j sit together.

Tables, not everybody can sit together

- You have M tables, each with a capacity of C.
- You want to assign attendees to tables in such a way that the total preference value is maximized.
- Decision variables: $x_{im} = 1$ if attendee i is assigned to table m, 0 otherwise.

Respect the capacity of the tables

$$\sum_{i=1}^{N} x_{im} \leq C \quad \forall m$$

Respect the restrictions

$$x_{im} + x_{jm} \geq 2 \times r_{ij} \times x_{im} \quad \forall i, j, m$$
$$x_{im} + x_{jm} \geq 2 \times r_{ij} \times x_{jm} \quad \forall i, j, m$$

You can only assign one person to one table

$$\sum_{m=1}^{M} x_{im} = 1 \quad \forall i$$

Objective Function

$$\text{Maximize } Z = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{m=1}^{M} p_{ij} \times x_{im} \times x_{jm}$$

What is the problem? The objective function is not linear. We need to linearize it to make it simpler.

Linearization

- Introduce binary decision variables y_{ijm} that are equal to 1 if attendees i and j are assigned to table m, 0 otherwise.
- The objective function becomes:

$$\text{Maximize } Z = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{m=1}^{M} p_{ij} \times y_{ijm}$$

Linking constraints

- The linking constraints ensure that y_{ijm} is equal to 1 if attendees i and j are assigned to table m.

$$y_{ijm} \geq x_{im} + x_{jm} - 1$$

$$y_{ijm} \leq x_{im}, \; y_{ijm} \leq x_{jm}$$

Full Model

$$\text{Maximize } Z = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{m=1}^{M} p_{ij} \times y_{ijm}$$

$$\sum_{i=1}^{N} x_{im} \leq C \quad \forall m$$

$$x_{im} + x_{jm} \geq 2 \times r_{ij} \quad \forall i, j, m$$

$$\sum_{m=1}^{M} x_{im} = 1 \quad \forall i$$

$$y_{ijm} \geq x_{im} + x_{jm} - 1 \quad \forall i, j, m$$

$$y_{ijm} \leq x_{im},\ y_{ijm} \leq x_{jm} \quad \forall i, j, m$$

$$x_{im} \in \{0, 1\} \quad \forall i, m$$

$$y_{ijm} \in \{0, 1\} \quad \forall i, j, m$$

See attached Excel file for the modelling of the constraints.

- Excel's solver is limited to 200 decision variables and 100 constraints. For larger problems, you can use Python to "pass" the problem to a solver. More on this in the following sessions (project overview).

Using Solver with VBA

Consider the following problem:

$$\begin{aligned} \max_{x_1,x_2} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 4 \\ & x_1 - x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

References to cells in Excel

- In cell `B1` you will input x_1.
- In cell `B2` you will input x_2.
- In cell `B3` you will input the objective function `=3*B1+2*B2`.
- In cell `B4` you will input the first constraint `=B1+B2`.
- In cell `B5` you will input the second constraint `=B1-B2`.

Solve it in VBA

```
Sub SolveLinearProgram()

    ' Clear any previous solver settings
    SolverReset

    ' Set the objective: Maximize Z = 3*X1 + 2*X2 (which is in B3)
    SolverOk SetCell:=Range("B3"), MaxMinVal:=1, ValueOf:=0, ByChange:=Range("B1:B2")

    ' Add the constraints:
    ' X1 + X2 <= 4 (which is in B4)
    SolverAdd CellRef:=Range("B4"), Relation:=1, FormulaText:=4
    ' X1 - X2 <= 1 (which is in B5)
    SolverAdd CellRef:=Range("B5"), Relation:=1, FormulaText:=1
    ' X1 >= 0
    SolverAdd CellRef:=Range("B1"), Relation:=3, FormulaText:=0
    ' X2 >= 0
    SolverAdd CellRef:=Range("B2"), Relation:=3, FormulaText:=0

    ' Solve the problem
    SolverSolve UserFinish:=True

    ' Keep the Solver solution
    SolverFinish KeepFinal:=1

End Sub
```

Explanation of the code

- `SolverReset`: Clears any existing solver settings.
  - `SolverOk`: Sets the objective function in cell B3 (the formula for Z). The argument `MaxMinVal:=1` indicates that we are maximizing the objective. `ByChange`: Refers to the decision variables x_1 and x_2 in the range `B1:B2`.
- `SolverAdd`: Adds constraints. For each constraint, we specify the cell reference, the type of relationship (Relation), and the target value (FormulaText). For example, `Relation:=1` corresponds to <=, and `Relation:=3` corresponds to >=.
- `SolverSolve`: Solves the linear program, and the argument `UserFinish:=True` means the solution will be completed without showing Solver's results dialog.
- `SolverFinish`: Keeps the solution in place after Solver completes.