Optimization using Solver

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Optimization

Notation and Definitions

- **Optimization**: The process of finding the minimum (or maximum) value of a function subject to a set of constraints.
- **Objective function**: The function to be minimized or maximized.
- **Decision variables**: The variables that can be adjusted to optimize the objective function.
- **Constraints**: The limitations on the decision variables.

Easy Optimization: Convex Optimization

Convex Optimization

- **Convex function**: A function that has a non-negative second derivative.
- **Convex set**: A set where the line segment between any two points in the set lies entirely within the set.
- Minimization of a convex function over a convex set is a convex optimization problem. These problems are easy to solve because they have a single global minimum.

Example: Linear Programming

Linear Programming

- Objective function: A linear function.
- Constraints: Linear inequalities.
- Decision variables: Continuous.
- **Optimization**: Minimize or maximize the objective function.

Linear Programming Problem: Investment Portfolio Optimization

You are managing an investment portfolio and have two options for investment:

- Investment A: This is a low-risk investment with a 5% return.
- Investment B: This is a higher-risk investment with a 10% return.

Your goal is to maximize the return on your investment, but there are constraints:

- The total amount available to invest is \$100,000.
- You decide that no more than 60% of the total investment should be in the higherrisk investment (Investment B).
- Due to diversification, you want to invest at least \$40,000 in Investment A.

Decision Variables:

Let:

- x_1 be the amount invested in Investment A.
- x_2 be the amount invested in Investment B.

Objective Function (Maximize total return): Maximize $Z = 0.05x_1 + 0.10x_2$

Constraints:

1. Total amount invested:

 $x_1+x_2\leq 100,000$

2. Limit on investment in higher-risk option:

 $x_2\leq 0.60(x_1+x_2)$

(This simplifies to $x_2 \leq 60,000$.)

3. Minimum investment in Investment A:

 $x_1 \geq 40,000$

4. Non-negativity constraints:

$$x_1\geq 0, \quad x_2\geq 0$$

Linear Programming Formulation:

Maximize $Z = 0.05x_1 + 0.10x_2$

Subject to:

 $egin{aligned} x_1+x_2 &\leq 100,000 \ x_2 &\leq 60,000 \ x_1 &\geq 40,000 \ x_1 &\geq 0, \quad x_2 &\geq 0 \end{aligned}$

Using Excel's Solver

- Installation: Go to File > Options > Add-ins .
- IN Manage: select Excel Add-ins and click Go . Install Solver Add-in .

Using Excel's Solver

• Define the parameters of the model in cells.

Asset	Return	Minimum Investment	Maximum Investment
Α	5%	40000	
В	10%		60000
		Total Investment	100000

^{Optimization resing Solver} **Define the objective function and constraints as formulas in Excel.**

	E	F	G	Н	I
5	Asset	Return	Minimum Investment	Maximum Investment	Investment
6	Α	0.05	40000		
7	В	0.1		60000	
8					
9			Total Investment	100000	

12	Obj	=F6*I6+F7*I7
13	C1	=16+17
14	C2	=16
15	C3	=17

Optim Solver: Data > Solver

	E	F	G	н	1
5	Asset	Return	Minimum Investment	Maximum Investment	Investment
6	Α	5%	40000		
7	В	10%		60000	
8					
9			Total Investment	100000	

12	Obj	0
13	C1	0
14	C2	0
15	C3	0

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		_		

Optime Constraints

Click Add to add the constraints.

	E	F	G	Н	I
5	Asset	Return	Minimum Investment	Maximum Investment	Investment
6	Α	5%	40000		
7	В	10%		60000	
8					
9			Total Investment	100000	



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	E	F	G	Н	I.
5	Asset	Return	Minimum Investment	Maximum Investment	Investment
6	A	5%	40000		
7	В	10%		60000	
8					
9			Total Investment	100000	



Add Constraint		\times
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	E	F	G	Н	I
5	Asset	Return	Minimum Investment	Maximum Investment	Investment
6	Α	5%	40000		
7	В	10%		60000	
8					
9			Total Investment	100000	

12	Obj	0
13	C1	0
14	C2	0
15	C3	0

\$F\$15] <=	57	1

- For bounds $x_1 \geq 0$ and $x_2 \geq 0$, we can use the Non-Negative option in Solver.
- The solving method can be set to Simplex LP (for linear programming problems).

Dotimizati	E	F	G	Н	I
5 סרוות ביות ביות ביות ביות ביות ביות ביות ב	Asset	Return	Minimum Investment	Maximum Investment	Investment
6	Α	5%	40000		40000
7	В	10%		60000	60000
8					
9			Total Investment	100000	

12	Obj	8000
13	C1	100000
14	C2	40000
15	C3	60000

olver Results		×
Solver found a solution. All Constraints and optim	ality	
conditions are satisfied.	Re <u>p</u> orts	
• Keep Solver Solution	Answer Sensitivity Limits	
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OK Cancel		Save Scenario
Solver found a solution. All Constraints and optima	lity conditions are satisfie	
When the GRG engine is used, Solver has found at is used, this means Solver has found a global optim	least a local optimal solu mal solution.	tion. When Simplex LP

Convex Optimization

- What if we also care about the risk of the investment?
- Volatility asset A: 2%
- Volatility asset B: 5%
- Correlation: 0.85
- Risk aversion: 1.0

The new objective function:

$$\text{Maximize } Z = 0.05 x_1 + 0.10 x_2 - \frac{1}{2} \times (0.02^2 \times x_1^2 + \frac{1}{2} \times 0.05^2 \times x_2^2 + 2 \times 0.02 \times 0.05 \times 0.85 \times x_1 \times x_2)$$

	0	P	Q	R	S	Т
5	Asset	Return	Volatility	Minimum Investment	Maximum Investment	Investment
6	А	0.05	0.02	40000		40000
7	В	0.1	0.05		60000	60000
8		Corr	0.85			
9		Gamma	1	Total Investment	100000	

12	Obj	=P6*T6+P7*T7-(Q9/2)*((T6^2)*Q6^2+(T7^2)*Q7^2+2*T6*T7*Q6*Q7*Q8)
13	C1	=T6+T7
14	C2	=T6
15	C3	=17

	0	Р	Q	R	S	Т
5	Asset	Return	Volatility	Minimum Investment	Maximum Investment	Investment
6	Α	<mark>5</mark> %	2%	40000		40000
7	В	10%	5%		60000	60000
8		Corr	0.85			
9		Gamma	1	Total Investment	100000	

12	Obj	-6852000
13	C1	100000
14	C2	40000
15	C3	60000

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S <u>u</u> bject to	the Constra	ints:				
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Solving M Select the Simplex e problems	Method e GRG Nonli engine for lii s that are no	near engine for near Solver Prol n-smooth.	r Solver Problems blems, and select	that are smo the Evolutior	oth nonlir nary engin	near. Select the LP e for Solver

	<u> </u>	1	¥	n	0	•
Optimizatio	Asset	Return	Volatility	Minimum Investment	Maximum Investment	Investment
6	A	5%	2%	40000		40000
7	В	10%	5%		60000	0
8		Corr	0.85			
9		Gamma	1	Total Investment	100000	

12	Obj	-318000
13	C1	40000
14	C2	40000
15	C3	0

onditions are satisfied.	Reports	
 Keep Solver Solution Restore Original Values 	Answer Sensitivity Limits	
Return to Solver Parameters Dialog	Outline Reports	

MIP (Mixed-Integer Programming)

- Convex constraints are not enough to represent all real-world problems.
- Many problems require integer or binary decision variables. These variables can represent integer quantities or **binary** decisions that must be made during the optimization.

First Example: Buying Shares

- You can only buy shares in whole numbers. (Although you can pool your money with others to buy a fraction of a share.)
- In the last example, assume company A's shares cost \$10 and company B's shares cost \$20. How would you modify the model?

Maximize $Z = 0.05 imes 10 imes n_1 + 0.10 imes 20 imes n_2$

Subject to:

$$egin{aligned} n_1 imes 10 + n_2 imes 20 &\leq 100,000 \ n_2 imes 20 &\leq 60,000 \ n_1 imes 10 &\geq 40,000 \ n_1 \in \mathbb{N}_+, \quad n_2 \in \mathbb{N}_+ \end{aligned}$$

	0	P	Q	R	S	Т		
5	Asset	Return	Volatility	Minimum Investment	Maximum Investment	Investment	Price	Shares
6	Α	0.05	0.02	40000		=AD6*AE6	10	4000
7	В	0.1	0.05		60000	=AD7*AE7	20	0
8		Corr	0.85					
9		Gamma	1	Total Investment	100000			

12	Obj	=Y6*AC6+Y7*AC7-(Z9/2)*((AC6^2)*Z6^2+(AC7^2)*Z7^2+2*AC6*AC7*Z6*Z7*Z8)
13	C1	=AC6+AC7
14	C2	=AC6
15	C3	=AC7

Se <u>t</u> Objective:		\$Y\$12		Î
To: O <u>M</u> ax	() Mi <u>n</u>	○ <u>V</u> alue Of:	0	
<u>By</u> Changing Varia	ble Cells:			
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Subject to the Cor	ostraints:			
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	0	Р	Q	R	S	T		
Optimizati <mark>ð</mark> ı	Asset :	Beturn	Volatility	Minimum Investment	Maximum Investment	Investment	Price	Shares
6	Α	5%	2%	40000		40000	10	4000
7	В	10%	5%		60000	0	20	0
8		Corr	0.85					
9		Gamma	1	Total Investment	100000			

Solver round a solution. An constraints and	optimarity
conditions are satisfied.	Reports
• Keep Solver Solution	Answer
O <u>R</u> estore Original Values	
Return to Solver Parameters Dialog	Outline Reports
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 12
 Obj
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 13
 C1
 40000

 14
 C2
 40000

 15
 C3
 0

Second Example: Binary Decision

- Binary decisions are often used in optimization problems where a decision must be made between two options.
- Example: You are a manufacturer and have two options for a new product line: Product A and Product B. You can only choose one of them. How would you model this in an optimization problem?

The KnapSack Problem

- You have a knapsack with a weight limit of 15 kg.
- You have the following items to choose from: A, B, C, D, E, F.
- Each item has a weight and a utility value for you.

E.g., A: 5 kg, 10 value.
B: 3 kg, 7 value.
C: 4 kg, 8 value.
D: 6 kg, 12 value.
E: 2 kg, 6 value.
F: 7 kg, 14 value.

The KnapSack Problem

- You want to maximize the total value of the items you can carry in your knapsack without exceeding the weight limit.
- How would you model this as an optimization problem?
- Introduce binary decision variables y_i for each item i.

$$i\in\{A,B,C,D,E,F\} \hspace{1em} y_i\in\{0,1\}$$

weights and utility

 w_i and v_i

Objective Function:

$$ext{Maximize } Z = \sum_{i \in \{A,B,C,D,E,F\}} v_i imes y_i$$

Subject to:

$$\sum_{i\in \{A,B,C,D,E,F\}} w_i imes y_i \leq 15
onumber \ y_i \in \{0,1\}$$

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Seat Assignment Problem

- You are organizing a wedding and have to assign seats to attendees. Some attendees have requested to sit next to each other, while others have requested to sit apart (restrictions).
- You think the wedding will be more enjoyable if people with similar interests sit together (preferences).

Seat Assignment Problem

Attendees = $i = 1, \dots, N$ Restrictions

$$r_{ij} = egin{cases} 1 & ext{if attendees } i ext{ and } j ext{ must sit together} \ 0 & ext{otherwise} \end{cases}$$

Preferences p_{ij} , the higher the value, the more fun the wedding will be if attendees i and j sit together.

Tables, not everybody can sit together

- You have M tables, each with a capacity of C.
- You want to assign attendees to tables in such a way that the total preference value is maximized.
- Decision variables: $x_{im} = 1$ if attendee i is assigned to table m, 0 otherwise.

Respect the capacity of the tables

$$\sum_{i=1}^N x_{im} \leq C \quad orall m$$

Respect the restrictions

$$egin{aligned} x_{im}+x_{jm}&\geq 2 imes r_{ij} imes x_{im} &orall i,j,m\ x_{im}+x_{jm}&\geq 2 imes r_{ij} imes x_{jm} &orall i,j,m \end{aligned}$$

You can only assign one person to one table

$$\sum_{m=1}^M x_{im} = 1 \quad orall i$$

Objective Function

$$ext{Maximize } Z = \sum_{i=1}^N \sum_{j=1}^N \sum_{m=1}^M p_{ij} imes x_{im} imes x_{jm}$$

What is the problem? The objective function is not linear. We need to linearize it to make it simpler.

Linearization

- Introduce binary decision variables y_{ijm} that are equal to 1 if attendees i and j are assigned to table m, 0 otherwise.
- The objective function becomes:

$$ext{Maximize} \ Z = \sum_{i=1}^N \sum_{j=1}^N \sum_{m=1}^M p_{ij} imes y_{ijm}$$

Linking constraints

• The linking constraints ensure that y_{ijm} is equal to 1 if attendees i and j are assigned to table m.

$$egin{aligned} y_{ijm} \geq x_{im} + x_{jm} - 1 \ y_{ijm} \leq x_{im}, \ y_{ijm} \leq x_{jm} \end{aligned}$$

Full Model

$$egin{aligned} ext{Maximize} \ Z &= \sum_{i=1}^N \sum_{j=1}^N \sum_{m=1}^M p_{ij} imes y_{ijm} \ & \sum_{i=1}^N x_{im} \leq C \quad orall m \ & x_{im} + x_{jm} \geq 2 imes r_{ij} \quad orall i, j, m \ & \sum_{m=1}^M x_{im} = 1 \quad orall i \ & y_{ijm} \geq x_{im} + x_{jm} - 1 \quad orall i, j, m \ & y_{ijm} \leq x_{im}, \ & y_{ijm} \leq x_{jm} \quad orall i, j, m \ & x_{im} \in \{0,1\} \quad orall i, m \ & y_{ijm} \in \{0,1\} \quad orall i, j, m \end{aligned}$$

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See attached Excel file for the modelling of the constraints.

• Excel's solver is limited to 200 decision variables and 100 constraints. For larger problems, you can use Python to "pass" the problem to a solver. More on this in the following sessions (project overview).

Using Solver with VBA

Consider the following problem:

$$egin{array}{ll} \max_{x_1,x_2} & 3x_1+2x_2 \ {
m subject to} & x_1+x_2 \leq 4 \ & x_1-x_2 \leq 1 \ & x_1,x_2 \geq 0 \end{array}$$

References to cells in Excel

- In cell B1 you will input x_1 .
- In cell B2 you will input x_2 .
- In cell B3 you will input the objective function =3*B1+2*B2.
- In cell B4 you will input the first constraint =B1+B2.
- In cell B5 you will input the second constraint =B1-B2.

Solve it in VBA

```
Sub SolveLinearProgram()
    ' Clear any previous solver settings
    SolverReset
    ' Set the objective: Maximize Z = 3*X1 + 2*X2 (which is in B3)
    SolverOk SetCell:=Range("B3"), MaxMinVal:=1, ValueOf:=0, ByChange:=Range("B1:B2")
    ' Add the constraints:
    ' X1 + X2 <= 4 (which is in B4)
    SolverAdd CellRef:=Range("B4"), Relation:=1, FormulaText:=4
    ' X1 - X2 <= 1 (which is in B5)
    SolverAdd CellRef:=Range("B5"), Relation:=1, FormulaText:=1
    ' X1 >= 0
    SolverAdd CellRef:=Range("B1"), Relation:=3, FormulaText:=0
    ' X2 >= 0
    SolverAdd CellRef:=Range("B2"), Relation:=3, FormulaText:=0
    ' Solve the problem
    SolverSolve UserFinish:=True
    ' Keep the Solver solution
    SolverFinish KeepFinal:=1
End Sub
```

Explanation of the code

- SolverReset : Clears any existing solver settings.
 - SolverOk : Sets the objective function in cell B3 (the formula for Z). The argument MaxMinVal:=1 indicates that we are maximizing the objective. ByChange : Refers to the decision variables x_1 and x_2 in the range B1:B2.
- SolverAdd : Adds constraints. For each constraint, we specify the cell reference, the type of relationship (Relation), and the target value (FormulaText). For example,
 Relation:=1 corresponds to <=, and Relation:=3 corresponds to >=.
- SolverSolve : Solves the linear program, and the argument UserFinish:=True means the solution will be completed without showing Solver's results dialog.
- SolverFinish : Keeps the solution in place after Solver completes.